

On Lerch's formula for the Fermat quotient

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January 15, 2013

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Abstract

This paper explores some previously-unrecognized consequences of Lerch's 1905 formula for the Fermat quotient, with special attention to the sums $s(k, N) = \sum_{j=\lfloor kp/N \rfloor + 1}^{\lfloor (k+1)p/N \rfloor} \frac{1}{j}$ which he introduced in this context. A generalization of his result is proved in the case of composite N , and a new proof given of a sharpened result by Skula (2008) [15] when N is even. We also sharpen the criteria given by Emma Lehmer in 1938 for a Wieferich prime to be simultaneously a Mirimanoff prime.

Keywords: Fermat quotient, Wieferich prime, Mirimanoff prime

1 Introduction

For the Fermat quotient $q_p(b) = (b^{p-1} - 1)/p \pmod{p}$ we employ wherever possible the briefer notation q_b . Henceforth all congruences are assumed to be mod p unless otherwise stated, and $\lfloor \cdot \rfloor$ signifies the greatest-integer function. The fact that Fermat quotients can be expressed as sums involving reciprocals of integers in $\{1, p-1\}$ was discovered in 1850 for the case $b = 2$ by Eisenstein, who gives $q_2 \equiv 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{(p-2)} - \frac{1}{(p-1)}$. Subsequent researches in this direction have focused on developing equivalent results entailing fewer terms. Later Wieferich and Mirimanoff demonstrated the connection of Fermat quotients with the first case of Fermat's Last Theorem (FLT). This connection retains its historical interest despite the complete proof of FLT by Wiles in 1995, and Skula's demonstration in 1992 [14] that the failure of the first case of FLT would imply the vanishing of many similar sums but with much smaller ranges, which cannot be evaluated in terms of Fermat quotients.

Lerch's remarkable formula of 1905 [12] is

$$N \cdot q_p(N) \equiv \sum_{k=1}^{N-1} k \cdot s(k, N), \quad (1)$$

where

$$s(k, N) = \sum_{j=\lfloor \frac{kp}{N} \rfloor + 1}^{\lfloor \frac{(k+1)p}{N} \rfloor} \frac{1}{j}.$$

For a detailed exposition in English, see Agoh et al. ([1], pp. 32–35). In the case of composite N , the left-hand side of (1) is evaluated using Eisenstein's logarithmic property $q(ab) = q(a) + q(b)$. The fact that the terms in $\{\frac{(p+1)}{2}, p-1\}$ are the mirror-image \pmod{p} of those in $\{1, \frac{(p-1)}{2}\}$ implies that $s(k, N) \equiv -s(N-1-k, N)$. For ease of comparison with previous literature, in our final results we usually restrict k so as to be less than $\frac{(p-1)}{2}$, or to be of a particular parity; but in the proofs we use whichever form seems more intelligible or expressive in the given situation.

As Lerch himself noted, the complementarity of the terms about the middle of the range $\{1, p-1\}$ leads to considerable simplification of (1) above, with $s(\frac{(N-1)}{2}, N)$ vanishing for odd N , so that

$$\begin{aligned} N \cdot q_p(N) \equiv & \\ - (N-1) \cdot s(0, N) - (N-3) \cdot s(1, N) - (N-5) \cdot s(2, N) - \dots - s(\frac{N}{2}-1, N) & \\ [N\text{even}] & \end{aligned}$$

$$\begin{aligned} N \cdot q_p(N) \equiv & \\ - (N-1) \cdot s(0, N) - (N-3) \cdot s(1, N) - (N-5) \cdot s(2, N) - \dots - 2 \cdot s(\frac{(N-1)}{2}-1, N) & \\ [N\text{odd}]. \quad (2) & \end{aligned}$$

These results for the Fermat quotient are so comprehensive as to subsume all those previously achieved. The cases $N = 2$ and $N = 4$ give, respectively, Glaisher's 1901 results ([7], pp. 21-22, 23) $s(0, 2) \equiv -2 \cdot q_2$ and $s(0, 4) \equiv -3 \cdot q_2$. Lerch himself ([12], p. 476, equations 14 and 15) pretty much explicitly writes out $s(0, 3) \equiv -\frac{3}{2} \cdot q_3$ and $2 \cdot s(0, 5) + s(1, 5) \equiv -\frac{5}{2} \cdot q_5$, correcting work of Sylvester ([25], pp. 161–162). Although Lerch could have used his formulae to evaluate $s(0, 6)$, it was left to Emma Lehmer in 1938 ([10], pp. 356ff) to point out that $s(0, 6) \equiv -2 \cdot q_2 - \frac{3}{2} \cdot q_3$, and we have not found in any publication prior to that of Granville and Sun in 1996 ([9], p. 136) an explicit statement that the two instances of $s(k, 12)$ lying at the center of the range $\{1, \frac{(p-1)}{2}\}$ can be evaluated by subtraction from known results, giving $s(2, 12) \equiv -q_2 + \frac{3}{2} \cdot q_3$, and $s(3, 12) \equiv 3 \cdot q_2 - \frac{3}{2} \cdot q_3$. However, it is convenient to group these instances among the “classical” results which were completely settled and systematized by Lerch's method. These are summarized in Table 1 below.

Dilcher & Skula ([4], p. 389] report numerical investigations of all possible values of $s(k, N)$ for the two values of p for which q_2 vanishes, namely 1093 and 3511. This occurs only for the “classical” cases, eliminating the possibility that any other sums could be simple multiples of q_2 . It therefore seems likely that our Theorems 1 and 2 comprise essentially all the linear relations which pertain among sums of Lerch's type.

2 Supplementary Notations

Certainly, not all sums figuring in the literature of Fermat quotients can be reduced to Lerch's type, least of all those containing the numbers of Bernoulli, Euler, Fibonacci, Lucas, or Pell. However, the sums studied here are the simplest representatives of an important family of interrelated sums whose other members we designate as follows:

$$\begin{aligned}
s'(k, N) &= \text{terms with odd denominators in } s(k, N) \\
&\equiv -\frac{1}{2} \cdot s(N-1-k, 2N) \equiv \frac{1}{2} \cdot s(N+k, 2N) \quad (3)
\end{aligned}$$

$$\begin{aligned}
s''(k, N) &= \text{terms with even denominators in } s(k, N) \equiv \frac{1}{2} \cdot s(k, 2N) \\
&\quad (4)
\end{aligned}$$

$$\begin{aligned}
s'''(k, N) &= \text{terms with denominators divisible by 3 in } s(k, N) \equiv \frac{1}{3} \cdot s(k, 3N) \\
&\quad (5)
\end{aligned}$$

$$\begin{aligned}
s^*(k, N) &= s''(k, N) - s'(k, N) \equiv \frac{1}{2} \cdot s(k, 2N) - \frac{1}{2} \cdot s(N+k, 2N) \\
&\quad (6)
\end{aligned}$$

$$\begin{aligned}
K(r, N) &= \text{terms in } s(0, 1) \text{ congruent to } rp \text{ mod } N \\
&\equiv \frac{1}{N} \cdot s(N-r, N) \equiv -\frac{1}{N} \cdot s(r-1, N) \quad (7)
\end{aligned}$$

$$\begin{aligned}
B(b, k, N) &\equiv \sum_{j=\lfloor \frac{kp}{N} \rfloor + 1}^{\lfloor \frac{(k+1)p}{N} \rfloor} \frac{b^j}{j}. \\
&\quad (8)
\end{aligned}$$

The evaluation of $s'(k, N)$ (3) is a simple consequence of the fact that a series expressed as a sum of terms of odd denominator may be condensed into a smaller range of unrestricted terms, as follows:

$$\sum_{j \text{ odd}}^n \frac{1}{j} \equiv \sum_{j \text{ odd}}^{2\lfloor (n-1)/2 \rfloor + 1} \frac{1}{j} \equiv - \sum_{j \text{ odd}}^{p-1} \frac{1}{j} \equiv -\frac{1}{2} \sum_{\lfloor (p-n+1)/2 \rfloor}^{(p-1)/2} \frac{1}{j}.$$

Conversely, a series expressed as a sum only of terms of even denominator may be simplified merely by factoring out the 2 in the denominator of its summand,

whence the formula for $s''(k, N)$ (4). The formula for $s^*(k, N)$ (6) follows immediately from these results. An important special case thereof, the simplification of which is effected using the Corollary below, is

$$s^*(0, N) \equiv -s(1, 2N), \quad (9)$$

while $s^*(0, 1) \equiv -s(1, 2) \equiv -2 \cdot q_2$ is the classic result from 1850 of Eisenstein ([5], p. 41), and $s^*(0, 2) \equiv -s(1, 4) \equiv -q_2$ is in effect given by Stern ([17], p. 185). Stern ([17], p. 184) also gives $s'(0, 1) \equiv q_2$, and Zhi-Hong Sun ([20], p. 288) extends this result to p^3 . It is easy to prove the following generalization of a statement in Sun and Sun ([21], p. 385):

$$s(k, N) + s^*(k, N) \equiv s(k, 2N). \quad (10)$$

$K(r, N)$ (7), the proof of the formula for which is deferred to (15) below, corresponds to the $K_m(s, p)$ of Zhi-Hong Sun [18] and to the $K_p(r, m)$ of Zhi-Wei Sun [23]; we however omit the parameter p to simplify the notation and make it more uniform with the rest. Obviously $K(0, 2) \equiv s''(0, 1)$ and $K(1, 2) \equiv s'(0, 1)$. Zhi-Hong Sun ([20], pp. 281, 286–288, 303) evaluates $K(1, 3)$, $K(1, 4)$, $K(1, 6)$, and $K(-1, 4) \bmod p^3$, and $K(-1, 3) \bmod p^2$, as well as some analogous sums over smaller ranges.

The function $s'''(k, N)$ (5) is not central to our argument, and we state without proof what are almost certainly the only results for it that can be expressed solely in terms of Fermat quotients:

$$s'''(0, 1) \equiv \frac{1}{3} \cdot s(0, 3) \equiv -\frac{1}{2} \cdot q_3 \quad (11)$$

$$s'''(0, 2) \equiv \frac{1}{3} \cdot s(0, 6) \equiv -\frac{2}{3} \cdot q_2 - \frac{1}{2} \cdot q_3 \quad (12)$$

$$s'''(1, 2) \equiv \frac{1}{3} \cdot s(1, 6) \equiv \frac{2}{3} \cdot q_2. \quad (13)$$

All of these have obvious relevance to Lehmer's problem. For an interesting variation $\bmod p^3$ see Zhi-Hong Sun ([20], p. 288).

The function $B(b, k, N)$ of (8), while lacking a generally-accepted notation, has been evaluated for many values of the parameters, usually with $b = \frac{1}{2}, 2, \frac{1}{3}$, or 3. Clearly when $b = 1$ or -1 , it includes the family of sums here designated $s(k, N)$ as a special case. It has a substantial if scattered literature beginning in 1912 with Bachmann [2], who gives $B(\frac{1}{2}, 0, 1) \equiv s(1, 4) \equiv q_2$, along with results from which the sums of the terms with odd or even denominators may be readily deduced. Zhi-Hong Sun ([20], p. 311, Remark 4.1) gives in effect $B(\frac{1}{2}, 0, 1) \equiv 4 \cdot K(-1, 4) \pmod{p^3}$, which on Sun's own showing is equivalent to $s(1, 4) \bmod p$, but not $\bmod p^2$. The study of the more difficult $B(\frac{1}{2}, 0, 2)$ seems to have begun in 1995 with Zhi-Wei Sun [22], who evaluates it (in effect) as $-s^*(0, 2) - s^*(2, 4)$. An easier proof is given in Shan & Wang [16], and finally Zhi-Hong Sun [19] expressed it in the more concise form $-s(2, 8)$, a simplification the possibility of which is implied by the definition of $-s^*(k, N)$ and by results

for $s(k, N)$ with $N = 8$ discussed below. $B(2, 0, 1)$, which is congruent mod p^2 but not mod p^3 to $-2s(1, 4) \equiv -2q_2$, is evaluated mod p^3 in Zhi-Hong Sun [20], Theorem 4.1, and, somewhat more simply, in Meštrović [13], Theorem 1.5. Despite the statements of both of these authors that this series is studied in Glaisher [8], we cannot find it mentioned therein. Z.-H. Sun ([18], pt. 2, Lemma 2.4) gives evaluations mod p , depending upon Lucas sequences, of $B(b, 0, 2)$ and $B(\frac{1}{b}, 0, 2)$ with b any integer.

3 A partial generalization of Lerch's formula for composite N

If N is prime, then Lerch's formula (1) appears to be the only linear relation which holds among his sums of order N . However, if N is composite, (1) can be manipulated in a way which does not appear to have been previously stated. For example, if N is even, we have

$$\begin{aligned} N \cdot q_p(N) - 2N \cdot q_p\left(\frac{N}{2}\right) &\equiv N \cdot q_p(2) \\ &\equiv -\{s(0, N) + s(2, N) + s(4, N) + \dots + s(N-2, N)\}. \end{aligned} \quad (14)$$

This coincides with a result given below, but because we want to prove a generalization of the formula (when N is composite) we cannot work directly from (1), but rather must develop a somewhat broader version of the underlying theory.

The essential idea of Lerch's formula is that for fixed N , every pair of values of $s(k, N)$ is connected with every other value through the sharing of multiples of each other's terms (mod p). Specifically, $s(k, N)$ by definition contains the reciprocals of all j in the range $\{\lfloor \frac{(k+1)p}{N} \rfloor, \lfloor \frac{kp}{N} \rfloor + 1\}$, and if M is any divisor of N (including N itself), and r the residue of k (mod M), then the corresponding values of Mj reduced mod p are distributed among the sums $s(rM, N)$ through $s(rM + M - 1, N)$ as follows:

| range of k | residue of Mj (mod p) |
|--|----------------------------|
| $\{0, \frac{N}{M} - 1\}$ | 0 |
| $\{\frac{N}{M}, 2 \cdot \frac{N}{M} - 1\}$ | $-p$ |
| $\{2 \cdot \frac{N}{M}, 3 \cdot \frac{N}{M} - 1\}$ | $-2p$ |
| \dots | \dots |
| $\{(M-1)\frac{N}{M}, N-1\}$ | $-(M-1)p$ |

That is, the values of Mj in the first row fall in $\{1, p-1\}$, while those in the second row fall in $\{p+1, 2p-1\}$ and must be reduced by p , those in the third row fall in $\{2p+1, 3p-1\}$ and must be reduced by $2p$, etc.

We can evaluate the sum of this two-dimensional array of terms in two different ways. First, we can collect the terms belonging to each value of Mj mod p as follows:

$$s(0, M) \equiv M \times \{\text{terms} \equiv 0 \pmod{M} \text{ in } s(0, 1)\} \equiv M \cdot K(0, M)$$

$$s(1, M) \equiv M \times \{\text{terms} \equiv -p \pmod{M} \text{ in } s(0, 1)\} \equiv M \cdot K(-1, M)$$

$$s(2, M) \equiv M \times \{\text{terms} \equiv -2p \pmod{M} \text{ in } s(0, 1)\} \equiv M \cdot K(-2, M)$$

.....

$$\begin{aligned} s(M-1, M) &\equiv M \times \{\text{terms} \equiv -(M-1)p \pmod{M} \text{ in } s(0, 1)\} \\ &\equiv M \cdot K(-(M-1), M). \quad (15) \end{aligned}$$

In other words $N \cdot K(r, N) \equiv s(-r, N) \equiv -s(r-1, N)$, a result which coincides with one of Zhi-Hong Sun ([18], pt. 3, p. 90, Corollary 3.1). Secondly, we can collect the terms belonging to each value of r , as follows:

Theorem 1

$$\begin{aligned} s(0, N) + s\left(\frac{N}{M}, N\right) + s\left(2 \cdot \frac{N}{M}, N\right) + \dots + s\left((M-1)\frac{N}{M}, N\right) \\ \equiv M\{s(0, N) + s(1, N) + s(2, N) + \dots + s(M-1, N)\} \\ \equiv M \cdot s(0, N/M) \end{aligned}$$

$$\begin{aligned} s(1, N) + s\left(\frac{N}{M} + 1, N\right) + s\left(2 \cdot \frac{N}{M} + 1, N\right) + \dots + s\left((M-1)\frac{N}{M} + 1, N\right) \\ \equiv M\{s(M, N) + s(M+1, N) + s(M+2, N) + \dots + s(2M-1, N)\} \\ \equiv M \cdot s(1, N/M) \end{aligned}$$

.....

$$\begin{aligned}
& s\left(\frac{N}{M} - 1, N\right) + s\left(2 \cdot \frac{N}{M} - 1, N\right) + s\left(3 \cdot \frac{N}{M} - 1, N\right) + \dots + s(N - 1, N) \\
& \equiv M\{s(N - M, N) + s(N - M + 1, N) + s(N - M + 2, N) + \dots + s(N - 1, N)\} \\
& \equiv M \cdot s\left(\frac{N}{M} - 1, N/M\right). \quad (16)
\end{aligned}$$

Now, while all the relations in (16) are valid, those in the bottom half of the display are by symmetry simply the negatives of those in the opposite rows in the top half, while if M is odd the statement made in the middle row is truistic because each side consists of sums of terms symmetrically distributed about the “center line” and thus vanishes mod p . Furthermore, these relations are degenerate when $M = N$, so M must be a proper divisor of N , and thus N cannot be prime.

The relationships expressed in (16) may be employed in two different ways. The simpler expressions at the far right, if tractable, will provide an explicit evaluation of the sums on the far left, although as we shall see in the cases of $N = 9$ and $N = 18$, it is not a foregone conclusion that the sums are better known for smaller values of N . Alternatively, the sums on the far left and in the middle are homogeneous, and the congruence always admits of some simplification, both by direct cancelation and by use of the rule $s(k, N) \equiv -s(N - k - 1, N)$.

We now examine some of the implications of (16) for particular cases of M , reserving a discussion of particular cases of N until the end.

3.1 The case $M = 2$

The case $M = 2$ of (16) is especially interesting. Let $x = N/M$; then:

$$\begin{aligned}
s(0, 2x) + s(x, 2x) & \equiv 2\{s(0, 2x) + s(1, 2x)\} & \equiv 2 \cdot s(0, x) \\
s(1, 2x) + s(x + 1, 2x) & \equiv 2\{s(2, 2x) + s(3, 2x)\} & \equiv 2 \cdot s(1, x) \\
\dots & \dots & \dots \\
s(x - 1, 2x) + s(2x - 1, 2x) & \equiv 2\{s(2x - 2, 2x) + s(2x - 1, 2x)\} & \equiv 2 \cdot s(x - 1, x),
\end{aligned} \tag{17}$$

with the usual redundancy in the second half of the display. The first row of (17) yields an important corollary which will be frequently invoked in the following pages:

Corollary 1

$$\begin{aligned}
s(0, 2x) + s(x, 2x) & \equiv 2\{s(0, 2x) + s(1, 2x)\} \\
& \Rightarrow s(0, 2x) + 2 \cdot s(1, 2x) - s(x, 2x) \equiv 0 \\
& \Rightarrow s(0, 2x) + 2 \cdot s(1, 2x) + s(x - 1, 2x) \equiv 0 \quad (18)
\end{aligned}$$

Among relations connecting the sums $s(k, N)$ for even N which do not depend on knowledge of the value of some sum with smaller N , this one is noteworthy in that it relates only three terms, whereas Lerch's (2) involves $\lfloor N/2 \rfloor$ terms. Other relations compare sums of two values of $s(k, N)$ for even N with one or two values of $s(k, N/2)$. In (18), subtract both sides from $2s(0, x) + s(\frac{x}{2} - 1, x)$. Then

$$s(0, 2x) + s(x - 2, 2x) \equiv 2s(0, x) + s(\frac{x}{2} - 1, x). \quad (19)$$

In (18), subtract $s(0, x)$ from both sides. Then

$$s(1, 2x) + s(x - 1, 2x) \equiv -s(0, x). \quad (20)$$

Also, using (18) and (6),

$$s^*(0, x) \equiv -s(1, 2x). \quad (21)$$

Corollary 2

Adding together the first two rows of (17), with the use of (6) and some manipulation we derive the relation:

$$s^*(0, x) + s^*(1, x) \equiv -s(1, x). \quad (22)$$

This is the explanation for the equivalence of some evaluations given by Zhi-Hong Sun ([18], pt. 3, Theorem 3.2, nos. 2 and 3).

3.2 The case $M = N/2$

In the first two rows of (16), let N be even, and $M = N/2$. Then

Corollary 3

$$\begin{aligned} & s(0, N) + s(2, N) + s(4, N) + \dots + s(N - 2, N) \\ & \equiv \frac{N}{2} \{s(0, N) + s(1, N) + s(2, N) + \dots + s(\frac{N}{2} - 1, N)\} \\ & \equiv \frac{N}{2} \cdot s(0, 2) \equiv -N \cdot q_2 \end{aligned} \quad (23a)$$

$$\begin{aligned} & s(1, N) + s(3, N) + s(5, N) + \dots + s(N - 1, N) \\ & \equiv \frac{N}{2} \{s(\frac{N}{2}, N) + s(\frac{N}{2} + 1, N) + s(\frac{N}{2} + 2, N) + \dots + s(N - 1, N)\} \\ & \equiv \frac{N}{2} \cdot s(1, 2) \equiv N \cdot q_2. \end{aligned} \quad (23b)$$

Now let $N = p - 1$; then the sums $s(k, p - 1)$ divide $\{1, p - 1\}$ into equal pieces of length 1, so that:

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{(p-2)} \equiv s'(0, 1) \equiv -\frac{1}{2} \cdot s(0, 2) \equiv q_2 \quad (24a)$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{(p-1)} \equiv s''(0, 1) \equiv -\frac{1}{2} \cdot s(1, 2) \equiv \frac{1}{2} \cdot s(0, 2) \equiv -q_2, \quad (24b)$$

supplying a slightly different proof of some important special cases of (3) and (4) already stated.

3.3 The case $M = 3$

The case $M = 3$ of the first few rows of (16) will likewise be needed below. Let $x = N/M$; then:

$$\begin{aligned} s(0, 3x) + s(x, 3x) + s(2x, 3x) \\ \equiv 3\{s(0, 3x) + s(1, 3x) + s(2, 3x)\} \\ \equiv 3 \cdot s(0, x) \end{aligned}$$

$$\begin{aligned} s(1, 3x) + s(x+1, 3x) + s(2x+1, 3x) \\ \equiv 3\{s(3, 3x) + s(4, 3x) + s(5, 3x)\} \\ \equiv 3 \cdot s(1, x). \quad (25) \end{aligned}$$

3.4 The case $M = N/3$

In the first three rows of (16), let N be divisible by 3, and $M = N/3$. Then

Corollary 4

$$\begin{aligned} s(0, N) + s(3, N) + s(6, N) + \dots + s(N-3, N) \\ \equiv \frac{N}{3} \{s(0, N) + s(1, N) + s(2, N) + \dots + s(\frac{N}{3} - 1, N)\} \\ \equiv \frac{N}{3} \cdot s(0, 3) \equiv -\frac{N}{2} \cdot q_3 \quad (26a) \end{aligned}$$

$$\begin{aligned} s(1, N) + s(4, N) + s(7, N) + \dots + s(N-2, N) \\ \equiv \frac{N}{3} \{s(\frac{N}{3}, N) + s(\frac{N}{3} + 1, N) + s(\frac{N}{3} + 2, N) + \dots + s(\frac{2N}{3} - 1, N)\} \\ \equiv \frac{N}{3} \cdot s(1, 3) \equiv 0 \quad (26b) \end{aligned}$$

$$\begin{aligned}
& s(2, N) + s(5, N) + s(8, N) + \dots + s(N-1, N) \\
& \equiv \frac{N}{3} \left\{ s\left(\frac{2N}{3}, N\right) + s\left(\frac{2N}{3} + 1, N\right) + s\left(\frac{2N}{3} + 2, N\right) + \dots + s(N-1, N) \right\} \\
& \equiv \frac{N}{3} \cdot s(2, 3) \equiv \frac{N}{2} \cdot q_3. \quad (26c)
\end{aligned}$$

Now let $N = p - 1$, so that $p \equiv 1 \pmod{3}$; then the sums $s(k, p-1)$ divide $\{1, p-1\}$ into equal pieces of length 1, so that:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{(p-3)} \equiv K(1, 3) \equiv \frac{1}{3} \cdot s(2, 3) \equiv \frac{1}{2} \cdot q_3 \quad (27a)$$

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \dots + \frac{1}{(p-2)} \equiv K(2, 3) \equiv \frac{1}{3} \cdot s(1, 3) \equiv 0 \quad (27b)$$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \dots + \frac{1}{(p-1)} \equiv K(0, 3) \equiv \frac{1}{3} \cdot s(0, 3) \equiv -\frac{1}{2} \cdot q_3. \quad (27c)$$

These formulae are an improvement upon some of Glaisher ([7], p. 18), where only the difference between the first and third rows is evaluated. However, Lerch ([12], p. 476) gives a result from which the first row is easily deduced, and Lehmer ([10], p. 356) gives an equivalent result with the modulus p^2 .

For $p \equiv 2 \pmod{3}$, we obtain a parallel set of relations using (7):

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{(p-1)} \equiv K(2, 3) \equiv \frac{1}{3} \cdot s(1, 3) \equiv 0 \quad (28a)$$

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \dots + \frac{1}{(p-3)} \equiv K(1, 3) \equiv \frac{1}{3} \cdot s(2, 3) \equiv \frac{1}{2} \cdot q_3 \quad (28b)$$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \dots + \frac{1}{(p-2)} \equiv K(0, 3) \equiv \frac{1}{3} \cdot s(0, 3) \equiv -\frac{1}{2} \cdot q_3. \quad (28c)$$

These likewise improve upon Glaisher ([7], p. 18), where only the difference between the second and third rows is evaluated. However, Lerch ([12], p. 476) gives a result from which the middle row is easily deduced. Lehmer ([10], p. 356) gives an equivalent result with the modulus p^2 .

A few additional formulae in a similar vein but involving only terms in the first half of the range $\{1, p-1\}$ follow easily from the foregoing developments. If $p \equiv 1 \pmod{6}$,

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{(p-5)/2} \equiv \frac{1}{3} \cdot s(4, 6) \equiv -\frac{2}{3} \cdot q_2 \quad (29a)$$

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \dots + \frac{1}{(p-3)/2} \equiv \frac{1}{3} \cdot s(2, 6) \equiv -\frac{2}{3} \cdot q_2 + \frac{1}{2} \cdot q_3 \quad (29b)$$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \dots + \frac{1}{(p-1)/2} \equiv \frac{1}{3} \cdot s(0, 6) \equiv -\frac{2}{3} \cdot q_2 - \frac{1}{2} \cdot q_3. \quad (29c)$$

If $p \equiv 5 \pmod{6}$,

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{(p-4)/2} \equiv \frac{1}{3} \cdot s(2, 6) \equiv -\frac{2}{3} \cdot q_2 + \frac{1}{2} \cdot q_3 \quad (30a)$$

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \dots + \frac{1}{(p-3)/2} \equiv \frac{1}{3} \cdot s(4, 6) \equiv -\frac{2}{3} \cdot q_2 \quad (30b)$$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \dots + \frac{1}{(p-2)/2} \equiv \frac{1}{3} \cdot s(0, 6) \equiv -\frac{2}{3} \cdot q_2 - \frac{1}{2} \cdot q_3. \quad (30c)$$

Lehmer ([10], p. 358) gives a result equivalent to the middle rows of these last groupings, with the modulus p^2 . Zhi-Hong Sun ([18], pt. 1, Corollary 1.12) gives the first two rows of each group.

4 The remaining classical formulae

For the sake of completing an inventory of the “classical” formulae, we state the following:

If $p \equiv 1 \pmod{4}$,

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{(p-4)} \equiv K(1, 4) \equiv \frac{1}{4} \cdot s(3, 4) \equiv \frac{3}{4} \cdot q_2 \quad (31a)$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{(p-3)} \equiv K(2, 4) \equiv \frac{1}{4} \cdot s(2, 4) \equiv -\frac{1}{4} \cdot q_2 \quad (31b)$$

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \dots + \frac{1}{(p-2)} \equiv K(3, 4) \equiv \frac{1}{4} \cdot s(1, 4) \equiv \frac{1}{4} \cdot q_2 \quad (31c)$$

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \dots + \frac{1}{(p-1)} \equiv K(0, 4) \equiv \frac{1}{4} \cdot s(0, 4) \equiv -\frac{3}{4} \cdot q_2, \quad (31d)$$

Stern ([17], p. 187) gives the first and third rows, and Glaisher ([7], pp. 22–23) gives the first three rows. Lehmer ([10], p. 358) gives a result equivalent to the first row, mod p^2 .

If $p \equiv 3 \pmod{4}$,

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{(p-2)} \equiv K(3, 4) \equiv \frac{1}{4} \cdot s(1, 4) \equiv \frac{1}{4} \cdot q_2 \quad (32a)$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \dots + \frac{1}{(p-1)} \equiv K(2, 4) \equiv \frac{1}{4} \cdot s(2, 4) \equiv -\frac{1}{4} \cdot q_2 \quad (32b)$$

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \dots + \frac{1}{(p-4)} \equiv K(1, 4) \equiv \frac{1}{4} \cdot s(3, 4) \equiv \frac{3}{4} \cdot q_2 \quad (32c)$$

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \dots + \frac{1}{(p-3)} \equiv K(0, 4) \equiv \frac{1}{4} \cdot s(0, 4) \equiv -\frac{3}{4} \cdot q_2, \quad (32d)$$

Stern ([17], p. 187) gives the first and third rows, while Glaisher ([7], pp. 22–23) gives the first row only. Lehmer ([10], p. 358) gives a result equivalent to the third row, mod p^2 .

If $p \equiv 1 \pmod{6}$,

$$\frac{1}{1} + \frac{1}{7} + \frac{1}{13} + \dots + \frac{1}{(p-6)} \equiv K(1, 6) \equiv \frac{1}{6} \cdot s(5, 6) \equiv \frac{1}{3} \cdot q_2 + \frac{1}{4} \cdot q_3 \quad (33a)$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{14} + \dots + \frac{1}{(p-5)} \equiv K(2, 6) \equiv \frac{1}{6} \cdot s(4, 6) \equiv -\frac{1}{3} \cdot q_2 \quad (33b)$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \dots + \frac{1}{(p-4)} \equiv K(3, 6) \equiv \frac{1}{6} \cdot s(3, 6) \equiv \frac{1}{3} \cdot q_2 - \frac{1}{4} \cdot q_3 \quad (33c)$$

$$\frac{1}{4} + \frac{1}{10} + \frac{1}{16} + \dots + \frac{1}{(p-3)} \equiv K(4, 6) \equiv \frac{1}{6} \cdot s(2, 6) \equiv -\frac{1}{3} \cdot q_2 + \frac{1}{4} \cdot q_3 \quad (33d)$$

$$\frac{1}{5} + \frac{1}{11} + \frac{1}{17} + \dots + \frac{1}{(p-2)} \equiv K(5, 6) \equiv \frac{1}{6} \cdot s(1, 6) \equiv \frac{1}{3} \cdot q_2 \quad (33e)$$

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \dots + \frac{1}{(p-1)} \equiv K(0, 6) \equiv \frac{1}{6} \cdot s(0, 6) \equiv -\frac{1}{3} \cdot q_2 - \frac{1}{4} \cdot q_3. \quad (33f)$$

Lehmer ([10], p. 358) gives a result equivalent to the first row, mod p^2 .

If $p \equiv 5 \pmod{6}$,

$$\frac{1}{1} + \frac{1}{7} + \frac{1}{13} + \dots + \frac{1}{(p-4)} \equiv K(5, 6) \equiv \frac{1}{6} \cdot s(1, 6) \equiv \frac{1}{3} \cdot q_2 \quad (34a)$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{14} + \dots + \frac{1}{(p-3)} \equiv K(4, 6) \equiv \frac{1}{6} \cdot s(2, 6) \equiv -\frac{1}{3} \cdot q_2 + \frac{1}{4} \cdot q_3 \quad (34b)$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \dots + \frac{1}{(p-2)} \equiv K(3, 6) \equiv \frac{1}{6} \cdot s(3, 6) \equiv \frac{1}{3} \cdot q_2 - \frac{1}{4} \cdot q_3 \quad (34c)$$

$$\frac{1}{4} + \frac{1}{10} + \frac{1}{16} + \dots + \frac{1}{(p-1)} \equiv K(2, 6) \equiv \frac{1}{6} \cdot s(4, 6) \equiv -\frac{1}{3} \cdot q_2 \quad (34d)$$

$$\frac{1}{5} + \frac{1}{11} + \frac{1}{17} + \dots + \frac{1}{(p-6)} \equiv K(1, 6) \equiv \frac{1}{6} \cdot s(5, 6) \equiv \frac{1}{3} \cdot q_2 + \frac{1}{4} \cdot q_3 \quad (34e)$$

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \dots + \frac{1}{(p-5)} \equiv K(0, 6) \equiv \frac{1}{6} \cdot s(0, 6) \equiv -\frac{1}{3} \cdot q_2 - \frac{1}{4} \cdot q_3. \quad (34f)$$

Lehmer ([10], p. 358) gives a result equivalent to the fifth row, mod p^2 .

5 Skula's sharpening of Lerch's formula when N is even

Up to this point, our results may be regarded as a fairly mild generalization of Lerch's formula. In this section, however, we give a new derivation of a transformation which yields significantly improved results. Consider both rows of (23a), and let:

a = terms in the first row with $k < N/2$
 b = terms in the first row with $k \geq N/2$
 c = terms in the second row with $k < N/2$
 d = terms in the second row with $k \geq N/2$.

Now $a + b$, $c + d$ are already defined in (23a), while by definition $a + c \equiv s(0, 2) \equiv -2 \cdot q_2$, $b + d \equiv s(1, 2) \equiv 2 \cdot q_2$, $a \equiv -d$, and $c \equiv -b$. Thus, we have enough information to solve a and b as follows, with a corresponding to row (a) and b to row (b):

Theorem 2

$$\begin{aligned}
 s(0, N) + s(2, N) + s(4, N) + \dots + s(2 \cdot \lfloor \frac{N-1}{4} \rfloor, N) \\
 \equiv \frac{(N+2)}{4} \cdot s(0, 2) \equiv -\frac{(N+2)}{2} \cdot q_2 \quad (35a)
 \end{aligned}$$

$$\begin{aligned}
 s(1, N) + s(3, N) + s(5, N) + \dots + s(2 \cdot \lfloor \frac{N-3}{4} \rfloor + 1, N) \\
 \equiv -\frac{(N-2)}{4} \cdot s(0, 2) \equiv \frac{(N-2)}{2} \cdot q_2. \quad (35b)
 \end{aligned}$$

It should be noted that Skula ([15], p. 8, Corollary 2.4) proved by a somewhat different technique a result equivalent to the second row; and as the sum of (35a) and (35b) is by definition $s(0, 2)$, the value of the first row is an obvious consequence of Skula's result. Nevertheless, we feel that this theorem warrants a closer look because much of its interest lies in the way the results generated by the two rows supplement one another. In the left-hand sides, the values of $s(k, N)$ are simply those with k of the appropriate parity and strictly less than $N/2$. When $N \equiv 0 \pmod{4}$, the number of terms in the left-hand sides of the two rows above is the same; when $N \equiv 2 \pmod{4}$, the number of terms in the left-hand side of the second row is one less than that in the first row. All this will be clearer if we write k in its even form throughout and dovetail the values produced by the two rows:

$$\begin{array}{ll}
 s(2, 4) & \equiv -q_2 \\
 s(0, 2) \equiv s(4, 6) & \equiv -2 \cdot q_2 \\
 s(0, 4) \equiv s(4, 8) + s(6, 8) & \equiv -3 \cdot q_2 \\
 s(0, 6) + s(2, 6) \equiv s(6, 10) + s(8, 10) & \equiv -4 \cdot q_2 \\
 s(0, 8) + s(2, 8) \equiv s(6, 12) + s(8, 12) + s(10, 12) & \equiv -5 \cdot q_2 \\
 s(0, 10) + s(2, 10) + s(4, 10) \equiv s(8, 14) + s(10, 14) + s(12, 14) & \equiv -6 \cdot q_2 \\
 s(0, 12) + s(2, 12) + s(4, 12) \equiv s(8, 16) + s(10, 16) + s(12, 16) + s(14, 16) & \equiv -7 \cdot q_2 \\
 \dots & \dots
 \end{array} \quad (36)$$

These conditions are much stronger than those of Lerch (2 or 14), and are particularly parsimonious in the cases where N is oddly even (6, 10, 14, etc.). The case of $N = 6$ was solved by Lehmer in 1938 [10], and that of $N = 10$ by Skula in 2008 [15] (see below). The cases of $N = 8$ and $N = 10$ can be proved directly from the evaluations in Zhi-Hong Sun ([18], pt. 3, Theorems 3.1 and 3.3. respectively).

Furthermore, if in (35a) and (35b) we let $N = p - 1$; then again the sums $s(k, p - 1)$ divide $0, p - 1$ into equal pieces of length 1, so that:

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{\{2 \cdot \lfloor \frac{(p-2)}{4} \rfloor + 1\}} \equiv s'(0, 2) \equiv \frac{1}{4} \cdot s(0, 2) \equiv -\frac{1}{2} \cdot q_2 \quad (37a)$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{\{2 \cdot \lfloor \frac{(p-4)}{4} \rfloor + 2\}} \equiv s''(0, 2) \equiv \frac{3}{4} \cdot s(0, 2) \equiv -\frac{3}{2} \cdot q_2. \quad (37b)$$

The first row of 37 supplies a new proof of a theorem of Glaisher ([7], p. 23, §40, where one of the versions of the formula is printed with a missing coefficient).

The fact that Theorem 2 results in subsets of $\frac{1}{4}$ of the terms in $\{1, p - 1\}$ which are evaluable in terms of Fermat quotients suggests the question of whether subsets of $\frac{1}{6}$ or even of $\frac{1}{12}$ of the terms might be amenable to such treatment, as is the case for some of the individual values of $s(k, N)$ in Table 1. However, attempts to isolate a collection of non-consecutive terms $s(k, N)$ with the k in arithmetic progression and comprising only $\frac{1}{6}$ of the terms in $\{1, p - 1\}$ reveal that the result is so evaluable only when the values of k span the entire range. Unlike the formulae leading to the solutions in (35a) and (35b), here the formulae relating the six pieces leave their values under-determined. Even should additional, undiscovered relations exist, the difficulty can be shown to be insurmountable in general. If we attempt to evaluate the sums of $s(k, N)$ for every third value before or after the midpoint $\frac{(p-1)}{2}$ we confront the sums $s(0, 12) + s(3, 12)$ and $s(6, 12) + s(9, 12) \equiv -s(5, 12) - s(2, 12)$, and conversely if we attempt to evaluate the sums for every second value within a range of length $\frac{(p-1)}{3}$ we confront the sums $s(0, 12) + s(2, 12)$, $s(4, 12) + s(6, 12) \equiv -2 \cdot s(0, 12) - 10 \cdot q_2 - \frac{9}{2} \cdot q_3$, and $s(8, 12) + s(10, 12) \equiv -s(3, 12) - s(1, 12)$. None of these results can be expressed solely in terms of Fermat quotients because each entails precisely one value of $s(k, 12)$ which cannot be so expressed (see Table 2).

6 Some consequences of our results for particular cases of N

In what follows, we have nothing to add to the results for $N = 1, 2, 3, 4, 6$ (all included in Table 1) as surveyed by such authors as Emma Lehmer [10] and Dilcher & Skula [4]. Rather, we shall elaborate upon the implications of our theorems for certain other values of N , especially with reference to the vanishing of q_2 or q_3 , and to the interesting question, considered by Lehmer, of whether

they can vanish simultaneously. As is well known, the failure of the first case of FLT would require the exponent to satisfying both the congruence $q_2 \equiv 0$ of Wieferich (only known solutions 1093, 3511) and the congruence $q_3 \equiv 0$ of Mirimanoff (only known solutions 11, 1006003). Whether it is possible for the same number to satisfy both congruences remains an open question, but we have been able to sharpen Lehmer's criteria somewhat.

First, however, we must briefly review progress in the determination of Lerch's sums made since his own paper of 1905. Lehmer ([10], p. 352) made the perspicacious observation that for results relating to q_2 which involve separating the terms in $\{1, p-1\}$ into ranges of 2, 3, 4, or 6 equal parts, these numbers "can be characterized by the fact that their totient does not exceed two." Setting aside the exceptional computable cases $s(2, 12)$ and $s(3, 12)$ previously noted, the significance of this distinction was revealed in 1991 when H.C. Williams ([27], p. 440) showed that the evaluations of $s(1, 5)$, $s(1, 8)$, and $s(1, 12)$ depend respectively upon the Lucas sequences $U_{p-(\frac{5}{p})}(1, -1)$, $U_{p-(\frac{2}{p})}(2, -1)$, and $U_{p-(\frac{3}{p})}(4, 1)$, the first and second of which correspond to the well-known Fibonacci numbers and Pell numbers; here (\cdot) is the Legendre symbol. We shall say more below about the evaluation of $s(k, 8)$ and $s(k, 12)$. Zhi-Hong Sun ([18], Corollary 2.4) evaluated $s^*(0, 9)$, equivalent to $-s(1, 18)$, in terms of q_2 and a complex recurrence relation, while interestingly, the larger ranges $s(k, 9)$ have still not been evaluated for any value of k . Sun & Sun ([21], p. 385) evaluate $s(0, 10)$ in terms of q_2 and Fibonacci numbers, and Zhi-Hong Sun presents a group of similar formulae involving our $K(r, 10)$, some also dependent on q_5 , from which the values of $s(k, 10)$ for other k can be derived ([18], Theorem 3.1); also expressions for $s(1, 10)$ and $s(2, 10)$ involving Lucas numbers (Corollary 1.11), and some formulae involving $s^*(k, 15)$ and a variant of our $K(r, 15)$ with terms of alternating sign (Theorem 3.2).

Zhi-Wei Sun ([24], p. 2216) points out that $s(k, N)$ can be obtained by subtraction from known results for certain values of k when $N = 24, 40, 60$ (*i.e.* when N has no prime-power divisors other than 2, 4, 8, 3, or 5). These all involve recurrence sequences such as the Fibonacci numbers, and cannot be evaluated solely in terms of Lerch's sums.

6.1 $N = 8$

In the first row of (17), set $x = 4$. Then $s(0, 8) + s(4, 8) \equiv 2\{s(0, 8) + s(1, 8)\} \equiv 2 \cdot s(0, 4) \equiv -6 \cdot q_2$ which implies $s(0, 8) + 2 \cdot s(1, 8) + s(3, 8) \equiv 0$. In the first row of (16), set $N = 8$ and $M = 4$, giving $s(0, 8) + s(2, 8) + s(4, 8) + s(6, 8) \equiv 4 \cdot s(0, 2)$. When $q_2 \equiv 0$, a straightforward calculation then gives

$$s(0, 8) \equiv -s(1, 8) \equiv -s(2, 8) \equiv s(3, 8),$$

and pairwise, each of these relations is a necessary and sufficient condition for the vanishing of q_2 . While not without theoretical interest, such conditions do not entail fewer terms than those involving $s(k, 4)$.

As to the actual values of $s(k, 8)$, Williams ([27], p. 440) evaluates the sum

$$U_{p-(\frac{2}{p})}(2, -1) \equiv s(1, 8) + s(2, 8),$$

where U is a Pell number. With the application of our Corollary 1, the right-hand side can be evaluated as $s(0, 2) - s(0, 8) - s(3, 8) \equiv -2 \cdot q_2 + 2s(1, 8)$, allowing the values of $s(k, 8)$ to be obtained for all k (see Table 2). These are also readily obtainable from the values of $K(r, 8)$ tabulated in Zhi-Hong Sun ([18], Theorem 3.3). As every value of $s(k, 8)$ entails a Pell number, there is no reason to expect that they should be expressible as simple multiples of q_2 , and the fact that they generally cannot be is proven by the following cases where they vanish while q_2 does not (the calculations have been extended to $p \leq 1431906$ without finding any further solution):

$$\begin{array}{ll} s(0, 8) & p = 269, 8573, 1300709 \\ s(1, 8) & p = 29 \\ s(2, 8) & p = 193 \\ s(3, 8) & p = 23, 56993. \end{array}$$

6.2 $N = 16$

In (16), let $M = 4$. Then

$$s(0, 16) + s(4, 16) + s(8, 16) + s(12, 16) \equiv 4s(0, 4). \quad (38)$$

In the first two rows of (17), let $x = 8$. Then

$$s(0, 16) + s(8, 16) \equiv 2s(0, 16) + s(1, 16) \equiv 2 \cdot s(0, 8) \quad (39a)$$

$$s(1, 16) + s(9, 16) \equiv 2s(2, 16) + s(3, 16) \equiv 2 \cdot s(1, 8). \quad (39b)$$

The first row implies $s(0, 16) + 2 \cdot s(1, 16) - s(8, 16) \equiv 0$. When $q_2 \equiv 0$, $2 \cdot s(0, 8) + 2 \cdot s(1, 8) \equiv 2 \cdot s(0, 4) \equiv 0$, so $s(0, 16) + s(8, 16) + s(1, 16) + s(9, 16) \equiv 0$. Adding these expressions gives $2 \cdot s(0, 16) + 3 \cdot s(1, 16) + s(9, 16) \equiv 0$, furnishing a criterion for the vanishing of q_2 which entails only $\frac{3}{16}$ of the terms in $\{1, p-1\}$, a slight improvement on that with $N = 8$. However, such an improvement does not continue for higher powers of 2, as the relationship to $s(k, 4)$ becomes too tenuous. Theorem 2 generates identities in four terms which vanish when $q_2 \equiv 0$.

6.3 $N = 12$

We begin by noting that the conditions $s(2, 12) \equiv -q_2 + \frac{3}{2} \cdot q_3 \equiv 0$ and $s(3, 12) \equiv 3 \cdot q_2 - \frac{3}{2} \cdot q_3 \equiv 0$, which Lehmer inexplicably overlooked, provide sharp necessary criteria for the simultaneous vanishing of q_2 and q_3 . Of all such criteria, these sums have the smallest ranges. As in the cases of the conditions $s(0, 6) \equiv -2 \cdot q_2 - \frac{3}{2} \cdot q_3 \equiv 0$ and $s(2, 6) \equiv -2 \cdot q_2 + \frac{3}{2} \cdot q_3 \equiv 0$, these criteria are certainly insufficient individually, as proven by the following cases where the

sums vanish although neither q_2 nor q_3 does (the calculations have been extended to $p \leq 300000$ without finding any further solution):

$$\begin{aligned} s(0, 6) & \quad p = 61 \\ s(2, 6) & \quad p = 73, 83 \\ s(2, 12) & \quad p = 179, 619, 17807 \\ s(3, 12) & \quad p = 250829. \end{aligned}$$

Obviously, however, since $s(2, 12) + s(3, 12) \equiv 2 \cdot q_2$, if these two sums vanish together then so does q_2 . Because so much is already known in the case $N = 12$, we shall only note further that taking $M = 2$ in (16) gives:

$$s(0, 12) + s(6, 12) \equiv 2\{s(0, 12) + s(1, 12)\} \equiv 2 \cdot s(0, 6) \equiv -4 \cdot q_2 - 3 \cdot q_3 \quad (40a)$$

$$s(1, 12) + s(7, 12) \equiv 2\{s(2, 12) + s(3, 12)\} \equiv 2 \cdot s(1, 6) \equiv 4 \cdot q_2. \quad (40b)$$

Here, the more interesting relation is the second one, which gives $s(1, 12) \equiv s(4, 12)$ as another necessary and sufficient condition for the vanishing of q_2 ; thus (truistically) the simultaneous vanishing of q_2 and q_3 would imply

$$s(0, 12) \equiv -s(1, 12) \equiv -s(4, 12) \equiv s(5, 12).$$

Frobenius in his paper of 1914 ([6], p. 676) gives precisely this condition as a prerequisite for the failure of the first case of FLT, but his proof is not via the theory of the Fermat quotient.

As to the actual values of $s(k, 12)$, Williams ([27], p. 440) evaluates the sum

$$U_{p-(\frac{2}{p})}(4, 1) \equiv s(1, 12) + s(2, 12) + s(3, 12) + s(4, 12),$$

where U is a term in the Lucas sequence 0, 1, 4, 15, 56, The right-hand side of this is equivalent to $s(0, 2) - s(0, 12) - s(5, 12) \equiv -2 \cdot q_2 + 2 \cdot s(1, 12)$, allowing the values of $s(k, 12)$ to be obtained for all k (see Table 3). In addition, an evaluation of $s(0, 12)$ can be recognized with some effort in Granville & Sun ([9], p. 119). For $N = 12$, only $s(2, 12)$, $s(3, 12)$, $s(8, 12)$, and $s(9, 12)$ can be evaluated solely in terms of Fermat quotients.

6.4 $N = 24$

As recognized by Zhi-Wei Sun ([24], p. 2216), $s(k, 24)$ can be explicitly evaluated by subtraction from known values when $k = 2, 3, 8, 9$, etc., since

$$s(2, 24) \equiv s(0, 8) - s(0, 12) \equiv s(1, 12) + s(2, 12) - s(1, 8) \quad (41a)$$

$$s(3, 24) \equiv s(0, 6) - s(0, 8) \equiv s(1, 8) - s(2, 12) \quad (41b)$$

$$s(8, 24) \equiv s(2, 6) - s(3, 8) \equiv s(2, 8) - s(3, 12) \quad (41c)$$

$$s(9, 24) \equiv s(3, 8) - s(5, 12) \equiv s(3, 12) + s(4, 12) - s(2, 8). \quad (41d)$$

In (17), let $x = 12$. Then the relations in the *third* and *fourth* rows are:

$$s(2, 24) + s(14, 24) \equiv 2\{s(4, 24) + s(5, 24)\} \equiv 2 \cdot s(2, 12) \quad (42a)$$

$$s(3, 24) + s(15, 24) \equiv 2\{s(6, 24) + s(7, 24)\} \equiv 2 \cdot s(3, 12), \quad (42b)$$

which will clearly vanish if it is possible for q_2 and q_3 to vanish simultaneously.

6.5 $N = 9$

From (25), $s(0, 9) + s(3, 9) + s(6, 9) \equiv 3\{s(0, 9) + s(1, 9) + s(2, 9)\}$ which implies $2 \cdot s(0, 9) + 3 \cdot s(1, 9) + 4 \cdot s(2, 9) - s(3, 9) \equiv 0$, the strongest relation produced by Lerch's theorem other than the ones depending on the relationship with $s(k, 3)$, including those given by 25 and other cases of (16) with $n = 9$, $M = 3$. Although $s(k, 9)$ has not been evaluated for any value of k , it is known that in general it cannot be expressed as a simple multiple of q_3 , as proven by the following cases (apart from the trivial one of $k = 4$) where it vanishes while q_3 does not (the calculations have been extended to $p \leq 1043300$ without finding any further solution):

$$\begin{aligned} s(0, 9) & \quad p = 677, 6691 \\ s(1, 9) & \quad p = 151, 457, 971, 1439, 12613 \\ s(2, 9) & \quad p = 241, 739, 37799 \\ s(3, 9) & \quad p = 97, 58193. \end{aligned}$$

6.6 $N = 18$

We cannot add much to the knowledge of this little-studied case, other than to point out that Corollary 1 gives

$$s(0, 18) + 2 \cdot s(1, 18) + s(8, 18) \equiv 0, \quad (43)$$

while the second row of (25) gives

$$s(1, 18) + s(7, 18) + s(13, 18) \equiv 6 \cdot q_2, \quad (44)$$

and the two rows of Theorem 2 give, respectively,

$$s(0, 18) + s(2, 18) + s(4, 18) + s(6, 18) + s(8, 18) \equiv -10 \cdot q_2 \quad (45a)$$

$$s(1, 18) + s(3, 18) + s(5, 18) + s(7, 18) \equiv -8 \cdot q_2. \quad (45b)$$

In view of the work of Zhi-Hong Sun ([18], part 2, Corollary 2.4) in which he in effect evaluates $s(1, 18)$ (apply our 21 to his result), there is no reason to expect that $s(k, 18)$ would vanish with q_2 or with q_3 . Not only does it fail to do so for the two known Wieferich primes and for the two known Mirimanoff primes, but with the possible (but unlikely) exceptions of $k = 0, 8$, it cannot in general be expressed as a simple multiple of either of these Fermat quotients, as proven by

the following cases where it vanishes while neither of them does (the calculations have been extended to $p \leq 3835989$ without finding any further solution):

| | |
|------------|--|
| $s(0, 18)$ | $p = \dots$ |
| $s(1, 18)$ | $p = 47, 1777, 217337$ |
| $s(2, 18)$ | $p = 167$ |
| $s(3, 18)$ | $p = 1171, 37783$ |
| $s(4, 18)$ | $p = 137, 251, 1087, 1301, 2111, 5749$ |
| $s(5, 18)$ | $p = 4177, 1581479$ |
| $s(6, 18)$ | $p = 108541$ |
| $s(7, 18)$ | $p = 149, 35267$ |
| $s(8, 18)$ | $p = \dots$ |

The apparent scarcity of zeroes of $s(0, 18)$ and $s(8, 18)$ against those for $s(1, 18)$ has no obvious explanation, as all three figure in the most restrictive relation known (43), and the first two are not known to be more highly constrained than $s(4, 18)$, which has more zeroes than any other value of $s(k, 18)$ in the range tested. Nor is it apparent why the distribution of zeroes in the tested range is so strikingly different from that of $N = 8$.

We believe Dilcher & Skula ([4], pp. 389–390) are in error when they state that there are zeros of $s(k, N)$ with $p < 2000$ for all values of N from 2 to 46 except 5.

6.7 $N = 5$

We shall not attempt to treat this case in any detail, for as previously noted, $s(k, 5)$ cannot generally be expressed in terms of Fermat quotients. We merely note that from Lerch's formula (1), if $q_5 \equiv 0$, then

$$5q_5 \equiv -4s(0, 5) - 2s(1, 5) \equiv 0,$$

which implies that

$$-2s(0, 5) - s(1, 5) \equiv 0.$$

From this condition and the obvious relationships between $s(k, 5)$ and $s(k, 10)$, it is straightforward to deduce that if $q_5 \equiv 0$, then

$$s(0, 5) \equiv s(4, 10)$$

and

$$s(1, 5) \equiv s(1, 10) \equiv -s(3, 10).$$

Furthermore, the results for $s(k, 10)$ discussed below show that if $q_2 \equiv q_5 \equiv 0$, then if either of the quantities $s(0, 5)$ or $s(1, 5)$ vanishes, so does the other.

6.8 $N = 10$

An early result for this value of N was given in terms of Fibonacci numbers by H.C. Williams ([26], p. 369), who evaluated $s^*(0, 5)$, equivalent to $-s(1, 10)$. Later, the work of Zhi-Hong Sun ([18], Corollary 1.11 and Theorem 3.1) provided explicit evaluations of $K(r, 10)$, and thus indirectly of $s(k, 10)$, for every value of k . Yet it is nonetheless interesting to consider the relations which pertain among these sums. Skula ([15], pp. 9-10) indeed made a special study of this case and gives $s(0, 10) + 2 \cdot s(1, 10) + s(4, 10) \equiv 0$, which corresponds to our Corollary 1 with $x = 5$, and $2 \cdot s(0, 10) + 3 \cdot s(1, 10) + 2 \cdot s(2, 10) + 3 \cdot s(3, 10) + 2 \cdot s(4, 10) \equiv 0$, which corresponds to our Theorem 1 with $M = 5$. These results can be easily read out of Sun's work (Theorem 3.1) although he does not state them explicitly.

Now Zhi-Hong Sun ([18], pt. 3, Theorem 3.2, nos. 3 and 5), proved two formulae which together yield the surprising result $s^*(1, 5) \equiv -s^*(0, 3)$. This apparently anomalous relationship between sums neither of whose N values divides the other becomes less mysterious when rewritten in the form $s(1, 10) + s(3, 10) \equiv -s(0, 6) - s(2, 6)$, which reveals it as a direct consequence of our Theorem 2. When $q_2 \equiv 0$ the right side of this congruence vanishes, giving $s(1, 10) + s(3, 10) \equiv 0$ as a remarkably compact condition, both necessary and sufficient, for the vanishing of q_2 . Comparison with Skula's second result then shows that when $q_2 \equiv 0$, we have also $s(0, 10) + s(2, 10) + s(4, 10) \equiv 0$. If in addition we assume that $q_5 \equiv 0$, the relations simplify as follows:

$$\begin{aligned} s(0, 10) &\equiv 3s(0, 5) \\ s(1, 10) &\equiv s(1, 5) \\ s(2, 10) &\equiv -4s(0, 5) \\ s(3, 10) &\equiv -s(1, 5) \\ s(4, 10) &\equiv s(0, 5) \end{aligned}$$

or, more succinctly,

$$4s(0, 10) \equiv -6s(1, 10) \equiv -3s(2, 10) \equiv 6s(3, 10) \equiv 12s(4, 10).$$

The derivations are not difficult, and we skip the details as these relations can be easily inferred from the explicit evaluations of $K(r, 10)$ given in [18].

7 Lehmer's problem

Although Emma Lehmer was not the first author to pose the question of whether q_2 and q_3 can vanish simultaneously, her 1938 paper remains the most important contribution to the subject. Indeed, there does not seem to have been much produced since, other than an heuristic argument against the possibility in Lenstra [11]. However, to the extent that Lehmer develops congruences for Fermat quotients to higher moduli, or derives expressions which cannot be expressed in terms of Lerch's sums, her work is supplemented by the extensive writings of Zhi-Hong Sun, notably by a major recent paper on Bernoulli and Euler numbers [20].

As previously noted, Lehmer overlooked the conditions involving $s(k, 12)$ discussed above, including the sharpest of all necessary criteria requiring only Fermat quotients, *i.e.* $s(2, 12) \equiv -q_2 + \frac{3}{2} \cdot q_3 \equiv 0$ and $s(3, 12) \equiv 3 \cdot q_2 - \frac{3}{2}q_3 \equiv 0$. We have given some comparable conditions involving $N = 24$. As to the vanishing of q_2 alone, see our (36), and of q_3 alone, our (26a); but although each of these results implies an infinite family of conditions, they do not appear to combine in any interesting way.

8 Remark on a result of Dilcher and Skula

Dilcher and Skula prove in [4] that the failure of the first case of Fermat's Last Theorem would imply

$$s(k, N) \equiv 0 \tag{46}$$

for all $N \leq 46$ and all $k < N$. Such a failure would of course entail the vanishing of q_2 . On that assumption, adding together (35a) and (35b) and cancelling the vanishing sums $s(0, N) + s(1, N)$, etc., we are left in the case $N \equiv 2 \pmod{4}$ with but a single term, $s(2 \cdot \lfloor \frac{N-1}{4} \rfloor, N) = s(\frac{N}{2} - 1, N) \equiv 0$, and applying (17), we find that all the values of $s(k, N)$ may be expressed as multiples of values of $s(k, \frac{N}{2})$. In other words, when the condition (46) is proved for an odd N and all $k < N$, the same condition is immediately proved for the case of $2N$. Thus, the result of Dilcher and Skula for $N \leq 46$ automatically extends to all oddly even $N \leq 90$.

This observation complements a result of Cikánek [3], in which it is shown that the failure of the first case of FLT implies (46) for all $N \leq 94$ and all $k < N$. Cikánek's proof requires the additional condition (stated in §3.4 of the paper) that $p > 5^{(N-1)^2(N-2)^2/4}$.

Table 1: Complete list of Lerch's sums (with $k < N/2$) which can be evaluated solely in terms of Fermat quotients

| | |
|------------|--|
| $s(0, 1)$ | 0 |
| $s(0, 2)$ | $-2 \cdot q_2$ |
| $s(0, 3)$ | $-\frac{3}{2} \cdot q_3$ |
| $s(1, 3)$ | 0 |
| $s(0, 4)$ | $-3 \cdot q_2$ |
| $s(1, 4)$ | q_2 |
| $s(0, 6)$ | $-2 \cdot q_2 - \frac{3}{2} \cdot q_3$ |
| $s(1, 6)$ | $2 \cdot q_2$ |
| $s(2, 6)$ | $-2 \cdot q_2 + \frac{3}{2} \cdot q_3$ |
| $s(2, 12)$ | $-q_2 + \frac{3}{2} \cdot q_3$ |
| $s(3, 12)$ | $3 \cdot q_2 - \frac{3}{2} \cdot q_3$ |

Table 2: Values of $s(k, 8)$, where $U_n = U_{p-(\frac{2}{p})}(2, -1)$ is a Pell number, and (\cdot) is the Legendre symbol; derived from Williams [27], p. 440

| | | | |
|------------|----------------|-----|---|
| $s(0, 12)$ | $-4 \cdot q_2$ | $-$ | $2 \cdot (\frac{2}{p}) \cdot \frac{U_n}{p}$ |
| $s(1, 12)$ | q_2 | $+$ | $2 \cdot (\frac{2}{p}) \cdot \frac{U_n}{p}$ |
| $s(2, 12)$ | $-q_2$ | $+$ | $2 \cdot (\frac{2}{p}) \cdot \frac{U_n}{p}$ |
| $s(3, 12)$ | $2 \cdot q_2$ | $-$ | $2 \cdot (\frac{2}{p}) \cdot \frac{U_n}{p}$ |

Table 3: Values of $s(k, 12)$, where $U_n = U_{p-(\frac{3}{p})}(4, 1)$ is a term in the Lucas sequence 0, 1, 4, 15, 56, ..., and (\cdot) is the Legendre symbol; derived from Williams [27], p. 440

| | | | | | |
|------------|----------------|-----|-------------------------|-----|---|
| $s(0, 12)$ | $-3 \cdot q_2$ | $-$ | $\frac{3}{2} \cdot q_3$ | $-$ | $3 \cdot (\frac{3}{p}) \cdot \frac{U_n}{p}$ |
| $s(1, 12)$ | q_2 | | | $+$ | $3 \cdot (\frac{3}{p}) \cdot \frac{U_n}{p}$ |
| $s(2, 12)$ | $-q_2$ | $+$ | $\frac{3}{2} \cdot q_3$ | | |
| $s(3, 12)$ | $3 \cdot q_2$ | $-$ | $\frac{3}{2} \cdot q_3$ | | |
| $s(4, 12)$ | $-3 \cdot q_2$ | | | $+$ | $3 \cdot (\frac{3}{p}) \cdot \frac{U_n}{p}$ |
| $s(5, 12)$ | q_2 | $+$ | $\frac{3}{2} \cdot q_3$ | $-$ | $3 \cdot (\frac{3}{p}) \cdot \frac{U_n}{p}$ |

References

- [1] Tagashi Agoh, Karl Dilcher, and Ladislav Skula, “Fermat Quotients for Composite Moduli,” *J. Number Theory* **66** (1997) 29–50.
- [2] P. Bachmann, “Über den Rest von $\frac{(2^{p-1}-1)}{p} \pmod{p}$,” *J. Reine Angew. Math.* **142** (1912) 41–50.
- [3] Petr Cikánek, “A special extension of Wieferich’s criterion,” *Math. Comp.* **62** (1994) 923–930.
- [4] Karl Dilcher & Ladislav Skula, “A New Criterion for the First Case of Fermat’s Last Theorem,” *Math. Comp.* **64** (1995) 363–392.
- [5] [G.] Eisenstein, “Eine neue Gattung zahlentheoretischer Funktionen, welche von zwei Elementen abhängen und durch gewisse lineare Funktional-Gleichungen definirt werden,” *Berichte Königl. Preuß. Akad. Wiss. Berlin* **15** (1850) 36–42.
- [6] G. Frobenius, “Über den Fermatschen Satz, III,” *Sitzungsber. Königl. Preuß. Akad. Wiss. Berlin* **33** (1914) 653–681.
- [7] J.W.L. Glaisher, “On the Residues of r^{p-1} to Modulus p^2 , p^3 , etc.,” *Q. J. Math. Oxford* **32** (1900–1901) 1–27.
- [8] J.W.L. Glaisher, “On the residues of the sums of products of the first $p-1$ numbers, and their powers, to modulus p^2 or p^3 ,” *Q. J. Math. Oxford* **31** (1899–1900) 321–353.
- [9] Andrew Granville & Zhi-Wei Sun, “Values of Bernoulli Polynomials,” *Pacific J. Math.* **172** (1996) 117–137.
- [10] Emma Lehmer. “On Congruences involving Bernoulli Numbers and the Quotients of Fermat and Wilson,” *Ann. of Math.* **39** (1938) 350–360.
- [11] H.W. Lenstra, “Miller’s Primality Test,” *Inform. Process. Lett.* **8** (1979) 86–88.
- [12] M. Lerch, “Zur Theorie des Fermatschen Quotienten...,” *Math. Ann.* **60** (1905) 471–490.
- [13] Romeo Meštrović, “An Extension of Sury’s Identity and related congruences,” *Bull. Aust. Math. Soc.* **85** (2012) 482–496.
- [14] Ladislav Skula, “Fermat’s Last Theorem and the Fermat Quotients,” *Comment. Math. Univ. St. Pauli* **41** (1992) 35–54.
- [15] Ladislav Skula, “A note on some relations among special sums of reciprocals modulo p ,” *Math. Slovaca* **58** (2008) 5–10.

- [16] Zun Shan & Edward T.H. Wang, “A simple proof of a curious congruence by Sun,” *Proc. Amer. Math. Soc.* **127** (1999) 1289–1291.
- [17] M. Stern, “Einige Bemerkungen über die Congruenz $\frac{(r^p-r)}{p} \equiv a \pmod{p}$,” *J. Reine Angew. Math.* **100** (1887) 182–188.
- [18] Zhi-Hong Sun, “[The] Combinatorial Sum $\sum_{k \equiv r \pmod{m}} \binom{n}{k}$ and its Applications in Number Theory, I” (in Chinese), *J. Nanjing Univ. Math. Biquarterly* **9** (1992): 227–240, **10** (1993): 205–118, **12** (1995): 90–102. A very full summary in English is available on the author’s website, at <http://www.hytc.cn/xsjl/szh/coms1.pdf>.
- [19] Zhi-Hong Sun, “Five congruences for primes,” *Fibonacci Quart.* **40** (2002) 345–351.
- [20] Zhi-Hong Sun, “Congruences involving Bernoulli and Euler numbers,” *J. Number Theory* **128** (2008) 280–312.
- [21] Zhi-Hong Sun & Zhi-Wei Sun, “Fibonacci Numbers and Fermat’s last theorem,” *Acta Arith.* **60** (1992) 371–388.
- [22] Zhi-Wei Sun, “A congruence for primes,” *Proc. Amer. Math. Soc.* **123** (1995) 1341–1346.
- [23] Zhi-Wei Sun, “On the Sum $\sum_{k \equiv r \pmod{m}} \binom{n}{k}$ and Related Congruences,” *Israel J. Math.* **128** (2002): 135–56.
- [24] Zhi-Wei Sun, “Binomial coefficients and quadratic fields,” *Proc. Amer. Math. Soc.* **134** (2006) 2213–2222.
- [25] J.J. Sylvester, “Sur une propriété des Nombres Premiers qui se rattache au Théorème de Fermat,” *C. R. Acad. Sci. Paris* **52** (1861) 161–163, 212–214 (correction), 307–308 (addendum), 817 (further correction). Reprinted in Sylvester’s *Collected Mathematical Papers*, 2:229–31 (with the briefer corrections incorporated, along with some silent editorial corrections), 232–233, 234–35, 241.
- [26] H.C. Williams, “A Note on the Fibonacci Quotient. . .,” *Canad. Math. Bull.* **25** (1982) 366–370.
- [27] H.C. Williams, “Some formulas concerning the fundamental unit of a real quadratic field,” *Discrete Math.* **92** (1991) 431–440.